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Multicriteria Engineering Optimization Problems: Statement, Solution and Applications

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Abstract The majority of engineering optimization problems (design, identification, design of controlled systems, optimization of large-scale systems, operational development of prototypes, and so on) are essentially multicriteria. The correct determination of the feasible solution set is a major challenge in engineering optimization problems. In order to construct the feasible solution set, a method called PSI (Parameter Space Investigation) has been created and successfully integrated into various fields of industry, science, and technology. Owing to the PSI method, it has become possible to formulate and solve a wide range of multicriteria optimization problems. In addition to giving an overview of the PSI method, this paper also describes the methods for approximation of the feasible and Pareto optimal solution sets, identification, decomposition, and aggregation of the large-scale systems.

Keywords Feasible solution set · Visualization tools · PSI method · MOVI software · Uniformly distributed sequences

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1 Introduction

When addressing different methods of optimization in order to solve engineering optimization problems, it is usually assumed that these problems are defined correctly. However, very often, this assumption is not true, especially if there is a set of contradictory criteria. Therefore, in the overwhelming majority of cases, an expert ends up solving ill-posed problems. In order to search for the best solutions, we first need to define the feasible solution set. This task generally presents serious, at times insurmountable difficulties.

In Sect. 2 of this paper, we present the generalized formulation of multicriteria optimization problems and the PSI method [1–7]. We show the necessity of the statement and solution of real-life optimization problems in an interactive mode. We also describe the approximation of the feasible solution set and Pareto optimal set and a related issue of regularization of the Pareto optimal set [6]. In Sect. 3, we present the statement and solution of the problem of vector identification of mathematical models, as well as the assessment of adequacy of these models to real objects [1–3]. We also consider the problem of operational development of a prototype. Section 4 describes multicriteria optimization problems for large-scale systems. Specifically, this section proposes constructing the feasible solution set of large-scale systems by means of decomposition and aggregation of its subsystems [2, 3]. In Sect. 5, we briefly discuss several of typical examples of real-life optimization problems that have been solved by the PSI method [1–6]. These are problems of multicriteria optimization [1–6] and vector identification [1]. The paper concludes with Sect. 6, which implies that definition of the feasible solution set is the "heart" of the search for optimal solutions of real-life optimization problems.

2 Multicriteria Design

2.1 Generalized Formulation of Multicriteria Optimization Problems

We discuss here a mathematical formulation that can be applied to the majority of engineering optimization problems [1–5]. Let us consider an object, whose operation is described by a system of equations (differential, algebraic, etc.) or whose performance criteria may be directly calculated. We assume that the system depends on r *design variables* or *parameters* (we use these two terms interchangeably) $\alpha_1, \dots, \alpha_r$ representing a point $\alpha = (\alpha_1, \dots, \alpha_r)$ of an r-dimensional space. It is common practice for α to appear in the aforementioned equations.

In the general case, when designing a machine, one has to take into account the design variable constraints, the functional constraints, and the criteria constraints.

The *design variable constraints* (constraints on the design variables) have the form

$$\alpha_j^* \le \alpha_j \le \alpha_j^{**}, \quad j = 1, \dots, r. \tag{1}$$

In the case of mechanical systems, the α_j represent the stiffness coefficients, the moments of inertia, masses, damping factors, geometric dimensions, etc.

First, we will consider these constraints, as well as the functional constraints to be rigidly fixed.

The *functional constraints* may be written as follows:

$$C_l^* \le f_l(\alpha) \le C_l^{**}, \quad l = 1, \dots, t, \tag{2}$$

where the *functional dependences* $f_l(\alpha)$ may be either functional, depending on the integral curves of the differential equations mentioned previously, or explicit functions of α (not related to the equations); and C_l^* and C_l^{**} are, respectively, the lower and the upper admissible values of the quantity $f_l(\alpha)$. The functional constraints can specify the range of allowable stresses in structural elements, the track gauge, etc.

There also exist particular performance criteria, such as productivity, materials consumption, and efficiency. It is desired that, all other things being equal, these criteria, denoted by $\Phi_\nu(\alpha)$, $\nu = 1, \dots, k$, take the extreme values. For simplicity, we assume that $\Phi_\nu(\alpha)$ are to be minimized.

The constraints (1) single out a parallelepiped Π in the r-dimensional design variable space (space of design variables). In turn, constraints (1) and (2) together define a certain subset G in Π, whose volume may be assumed to be positive without the loss of generality.

In order to avoid situations in which the expert regards the values of some criteria as unacceptable, we introduce the criteria constraints

$$\Phi_\nu(\alpha) \le \Phi_\nu^{**}, \quad \nu = 1, \dots, k, \tag{3}$$

where Φ_ν^{**} is the worst value of the criterion $\Phi_\nu(\alpha)$ to which the expert may agree. The choice of Φ_ν^{**} is discussed in what follows.

The criteria constraints differ from the functional constraints in that the former are determined when solving a problem and, as a rule, are repeatedly revised. Hence, unlike C_l^* and C_l^{**}, reasonable values of Φ_ν^{**} cannot be chosen before solving the problem.

Constraints (1)–(3) define the feasible solution set D, i.e., the set of design solutions α^i that satisfy the constraints, and hence, $D \subset G \subset \Pi$.

If the functions $f_l(\alpha)$ and $\Phi_\nu(\alpha)$ are continuous in Π, then the sets G and D are closed. Let us formulate one of the basic problems of multicriteria optimization. It is necessary to find such a set $P \subset D$ for which

$$\Phi(P) = \min_{\alpha \in D} \Phi(\alpha), \tag{4}$$

where $\Phi(\alpha) = (\Phi_1(\alpha), \dots, \Phi_k(\alpha))$ is the criterion vector and P is the Pareto optimal set.

In other words, a point, $\alpha^0 \in D$, is called a Pareto optimal point iff there exists no point $\alpha \in D$ such that $\Phi_\nu(\alpha) \le \Phi_\nu(\alpha^0)$ for all $\nu = 1, \dots, k$ and $\Phi_{\nu_0}(\alpha) < \Phi_{\nu_0}(\alpha^0)$ for at least one $\nu_0 \in \{1, \dots, k\}$. A set $P \subset D$ is called the Pareto optimal set iff it consists of Pareto optimal points.

The Pareto optimal set plays an important role in vector optimization problems, because it can be analyzed relatively easier than the feasible solution set and because, by definition, the optimal vector always belongs to the Pareto optimal set, irrespective of the system of preferences used by the expert for comparing vectors belonging to the feasible solution set.

It is worth noting that in the class of problems considered in this paper, the feasible solution set does not contain many solutions due to the rigid functional and criteria constraints. As a result, Pareto optimal set often contains very few solutions. Thus, we approach these problems by constructing the set of Pareto optimal solutions. Alternatively, there are some situations, when one or few Pareto optimal solutions can be obtained by optimizing some scalar function(s) and there may be no need for explicit construction of the Pareto optimal set (e.g., in bilevel problems). The latter problems are not considered here and are described in depth in [8].

2.2 Some Features of Engineering Optimization Problems

The engineering optimization problems considered in this work have the following common features:

(1) The problems are essentially multicriteria ones.
(2) Determination of the feasible solution set is one of the essential issues of the analysis of engineering problems. The construction of this set is an important step in the formulation and solution of such problems.
(3) The formulation and solution of the problem make up a single process. Customarily, the expert first formulates a problem and then a computer is employed to solve it. However, in the case under consideration this approach is unsuitable. The feasible solution set may be obtained only in the process of solution; therefore, the problems should be formulated and solved in an interactive mode. The analysis of the feasible set is of importance for the experts.
(4) As a rule, mathematical models are complicated systems of equations (including differential equations) that may be linear or nonlinear, deterministic or stochastic, with distributed or lumped parameters. The coefficients of these equations are parameters that can be varied within certain limits. Criteria are functions of these parameters. That is why, we deal with two spaces: "parameter space" (or we call it equivalently "design variable space") and its image, "criteria space". The latter space is often also called "objective space". The search for feasible solutions in these problems is associated with the need to investigate the parameter and parameter space.
(5) Very often, the experts do not encounter serious difficulties in analyzing the feasible solution set and the optimal set and in choosing the most preferred solution. They have a sufficiently well-defined system of preferences. Moreover, the aforementioned sets usually contain a small number of elements [1, 2, 9, 10].

To formulate and solve engineering optimization problems, the Parameter Space Investigation (PSI) method has been developed. A systematic and comprehensive description of the method can be found in [1–6].

2.3 Search in Multi-dimensional Domains by Using Uniformly Distributed Sequences

The features of the problems under consideration make it necessary to represent the vectors α by points of uniformly distributed sequences in the space of design variables [1, 2, 5]. In the following, we briefly consider this issue. H. Weyl was the first to give the definition of uniformity [11].

Let us consider a sequence of points $P_1, P_2, \dots, P_i, \dots$ belonging to a unit r-dimensional cube K^r. By G we denote an arbitrary domain in K^r, and by $S_N(G)$ the number of points P_i belonging to $G(l \le i \le N)$. A sequence (P_i) is called uniformly distributed in K^r, iff

$$\lim_{N\to\infty} \frac{S_N(G)}{N} = V_G, \tag{5}$$

where $V(G)$ is the volume of the r-dimensional domain G. If, instead of the unit cube, a parallelepiped Π is considered, then the right-hand side of (5) transforms into $V(G)/V(\Pi)$.

The intuitive meaning of the definition is the following: for large values of N, the number of points of a given sequence belonging to an arbitrary domain G is proportional to the volume $V(G)$, i.e. $S_N(G) \sim NV(G)$.

For many applied problems the following situation is typical. There exists a multi-dimensional domain in which a function (or a system of functions) is considered, whose values may be calculated at certain points. However, there is no sufficient information on the properties of the functions (criteria) under consideration, for example, about the order of their smoothness. In these cases, it is not possible to use the methods of searching for the optimal solutions, such as the gradient methods. Therefore, our approach consists of computing the performance criteria for sets of parameters (points), which are uniformly distributed in the domain of these criteria. Then, owing to the continuity of the criteria, which is typically the case, we will have information on the criteria in a neighborhood of each of these points. If there are enough points to cover, with their neighborhoods, all area under investigation (with satisfactory degree of approximation), then we will have full information on the values of criteria, and will be able to optimize them. If, when solving a practical problem, it is possible to calculate the system at N points, then (on the basis of the above) it is important that these points be distributed as uniformly as possible. Then the issue of choosing the best of all known uniformly distributed sequences can arise.

Prior to solving a specific problem, one cannot say with certainty which of the uniformly distributed sequences are most suitable. *Much depends on the behavior of the criteria, the form of the functional and the design variable constraints, and the geometry of the feasible solution set*. As was already mentioned, when solving practical problems, it is possible to compute the performance criteria only for a limited number of points defined by specific features of the problem. Therefore, the uniformity of an initial segment of the sequence is important. However, for small lengths of this segment, the property of uniformity of the sequence under consideration is not always apparent. The last circumstance complicates the effective utilization of one or another sequence. Therefore, when solving practical problems, any of the uniformly distributed sequences that are available today can be used with approximately equal efficiency. In this connection, it is appropriate to mention the works by J.H. Halton [12], J.M. Hammersley [13], E. Hlawka and R. Taschner [14], H. Faure [15], H. Niederreiter [16], J. Dick and H. Niederreiter [17], as well as the so-called LP_τ sequences [5] and new P_τ nets [2]. In all examples presented in this paper, points $Q_1, Q_2, \dots, Q_i, \dots$ of the LP_τ sequences were used.

2.4 The Parameter Space Investigation Method for Formulating and Solving Engineering Optimization Problems

Construction of the feasible solution set is essential for problems of this class. The feasible set, in its turn, depends on the correct definition of the criteria constraints Φ_ν^{**}. The values of these constraints must be a maximum over all possible feasible values of the respective criteria (since we seek to minimize criteria). Otherwise, since the criteria may be contradictory, and many feasible solutions may be lost. Thus, the solution of a multicriteria optimization problem to considerable extent is reduced to the correct definition of Φ_ν^{**}. Now we briefly proceed by describing the Parameter Space Investigation method, which allows correct determination of Φ_ν^{**} and, hence, of the feasible solution set.

Earlier we formulated the problem of multicriteria optimization and defined the feasible solution set D, which is constructed using the values of Φ_ν^{**}, $\nu = 1, \dots, k$ and some other constraints.

The PSI method involves the following three stages [1–6].

Stage 1 For each of the N points of the sequence uniformly distributed in Π, we calculate the system to be designed and check the functional constraints (2). If they hold, then the point $\alpha = \alpha^i$ is selected as a trial point in G and all $\Phi_\nu(\alpha)$ are calculated; otherwise, the point $\alpha = \alpha^i$ is discarded.[1] Then the test table is formed for each criterion. The values $\Phi_\nu(\alpha^1), \dots, \Phi_\nu(\alpha^N)$ in this table are arranged in the increasing order (if we seek to minimize $\Phi_\nu(\alpha)$).

Stage 2 This stage assumes interaction with an expert. Looking consecutively at each of the tables, the expert should define the greatest feasible value Φ_ν^{**} for each criterion.

Stage 3 Given the constraints Φ_ν^{**}, we construct the set D. By calculating the values of all criteria at the points $\alpha^{i_1}, \dots, \alpha^{i_s}$, one can readily verify if, among these points, there is at least one that led to satisfaction of all inequalities (3). All such points α^{i_j} form the feasible set D, and the problem (4) becomes solvable. Let the feasible set and the Pareto optimal set be constructed. Suppose that the expert found it possible to choose a new constraint $\overline{\Phi}_\nu^{**} > \Phi_\nu^{**}$ for some criterion. This is the case when Φ_ν^{**} is not the maximum feasible constraint. Since we already have computed the test table, we can readily construct the new feasible set and the new Pareto optimal set that correspond to all constraints $\overline{\Phi}_\nu^{**}$. This would require only calculations for the new values for the criteria. If the updated Pareto optimal set contains solutions that substantially improve the solutions from the Pareto optimal set constructed previously (for constraint Φ_ν^{**}), then the expert, based on the analysis of the results obtained, makes a final decision whether it is worth adopting the new values or retaining the old ones for the criterion constraints.

In practice, the expert imposes the criteria constraints in order to improve a prototype with respect to all criteria simultaneously. If it is impossible, he improves a prototype with respect to the most important criteria. In the process of dialogues with

[1] After the analysis of the test tables and the correction of the functional constraints, some of the unfeasible solutions can become feasible ones [1, 7].

a computer, the expert repeatedly revises the criteria constraints and carries out the multicriteria analysis. The PSI method gives the expert valuable information on the advisability of revising various criteria constraints with the aim of improving the basic criteria. The expert sees what *price one pays for making concessions in various criteria, i.e., what one loses and what one gains*.

After analyzing P, the expert finds the most preferred solution $\Phi(\alpha^0)$. As already noted, for the problems under consideration, experts do not have serious difficulties in analyzing the Pareto optimal set and in choosing the most preferred solution. Thus, the PSI method has proved to be a very convenient and effective tool for the expert, primarily because this method can be directly used for the statement and solution of the problem in an interactive mode.

2.5 The Complexity of the Investigation

The property of uniform distribution of points implies that $\gamma = V(D)/V(\Pi) \approx N_D/N$ for sufficiently large N. Here N is the number of points $\alpha^i \in \Pi$; N_D is the number of points that have entered the feasible solution set; $V(D)$ is the volume of the feasible solution set; $V(\Pi)$ is the volume of the parallelepiped Π. The ratio of the volumes γ in the Monte Carlo Theory is called the selection efficiency. For many engineering problems, $\gamma \ll 0.001$; therefore, the search for feasible solutions is like looking for a "needle in a haystack"; e.g., see [1, 2, 5, 9, 18]. In fact, γ characterizes the complexity of solving problems. Very often, after correction of constraints, the value of γ vastly increases. The number of tests in the real-life problems strongly depends on how close to "correct" the initial statement of a problem is.

When the PSI method is used, the majority of computer time is spent on determining the feasible solution set and the correction of the problem statement that eventually leads to obtaining the justified optimal solutions. The PSI method has proved to be a very convenient and effective tool for the expert, primarily because this method can be used for the statement and solution of the problem in an interactive mode.

2.6 "Soft" Functional Constraints and Pseudo-Criteria

For many practical problems, 'good' solutions that lie slightly beyond the limits imposed by the constraints can be found [1, 2, 5]. If the expert is informed of this, in some cases he will be ready to modify the constraints, so that the 'good' solutions will belong to the feasible solution set. The question is how to obtain such information.

Instead of the function $f_l(\alpha)$, whose constraints C_l^* and C_l^{**} are not rigid (i.e., are "soft"), we introduce an additional criterion $\Phi_{k+l}(\alpha) = f_l(\alpha)$, which we will call a pseudo-criterion. However, to find the value of Φ_{k+l}^{**} one has to compile a test table containing $\Phi_{k+l}(\alpha)$. By using the aforementioned algorithm, together with the new test table, one can define Φ_{k+l}^{**} in a way that prevents the loss of interesting solutions.

Strange as it may seem, when solving single-criterion engineering problems involving soft functional constraints, one has to use multicriteria formulations in order to find the feasible solution set. This is because Φ_ν^{**} may be determined correctly only upon analyzing the test table. In the general case, in solving a problem with soft functional constraints, one has to find the set D taking all performance criteria into account, the functions $f_l(\alpha)$ being considered as pseudo-criteria. In other words, one has to solve the problem with the constraints

$$\Phi_\nu(\alpha) \le \Phi_\nu^{**}, \quad \nu = 1, \dots, k, k+1, \dots, n.$$

It has already been mentioned that, in order to avoid 'multicriteriality', attempts were made to transform all criteria but one into functional dependences with constraints of the form (2). It is clear that one cannot proceed in this way, because it can lead to a considerable reduction in the feasible solution set. Whenever possible, the expert has to do just the opposite, viz., to transform the functional dependences into pseudo-criteria and then reduce the problem solution to the analysis of the test table.

The possibility of using pseudo-criteria is an important advantage of the PSI method. First, in many cases this allows us to choose functional and design variable constraints justifiably rather than arbitrarily. Second, as the number of constraints decreases, the volume of the domain G increases.

Finally, after the test tables that contain both criteria and pseudo-criteria have been constructed, only performance criteria are used in searching for the Pareto optimal solutions.

2.7 MOVI Software System and Visualization Tools

The test tables are the basis of the PSI method and MOVI software system that we have co-authored [7, 9, 19, 20]. Visualization tools that help identify and analyze the feasible solution set play an important role in MOVI. First of all these are design variable histograms (histograms of the distribution of the feasible solution set), criteria histograms, the table of functional failures, and graphs "criteria versus design variables" and "criteria versus criteria". Using these tools interactively, the expert virtually always corrects the initial statement of the problem, including constraints and the mathematical model itself.

2.8 Approximation of the Feasible Set

We have introduced the notion of a feasible solution in the multicriteria optimization problem. The algorithm discussed above allows simple and efficient identification and selection of feasible points from the design variable space. However, the following question arises: how can one use the algorithm to construct a feasible solution set D with a given accuracy? The feasible set is constructed by singling out a subset of $\Phi(D)$ that approximates any value of each criterion in region $\Phi(D)$ with a predetermined accuracy [6].

Let ε_ν be an admissible (in the expert's opinion) error in criterion Φ_ν. By ε we denote the error set $\{\varepsilon_\nu\}$, $\nu = 1, \dots, k$. We will say that region $\Phi(D)$ is approximated by a finite set $\Phi(D_\varepsilon)$ with an accuracy up to the set ε, if for any vector $\alpha \in D$, there can be found a vector $\beta \in D_\varepsilon$ such that

$$\left|\Phi_\nu(\alpha) - \Phi_\nu(\beta)\right| \le \varepsilon_\nu, \quad \nu = 1, \dots, k.$$

We assume that the functions we shall be operating with are continuous and satisfy the Lipschitz condition (L) formulated as follows: for all vectors α and β belonging to the domain of definition of the criterion Φ_ν, there exists a number L_ν such that

$$\left|\Phi_\nu(\alpha) - \Phi_\nu(\beta)\right| \leq L_\nu \max_j |\alpha_j - \beta_j|.$$

It follows that there exists L'_ν such that

$$\left|\Phi_\nu(\alpha) - \Phi_\nu(\beta)\right| \leq L'_\nu \sum_{j=1}^{r} |\alpha_j - \beta_j|.$$

We will say that a function $\Phi_\nu(\alpha)$ satisfies the special Lipschitz condition (*SL*) iff for all vectors α and β there exist numbers L_ν^j, $j = 1, \dots, r$ such that

$$\left|\Phi_\nu(\alpha) - \Phi_\nu(\beta)\right| \leq \sum_{j=1}^{r} L_\nu^j |\alpha_j - \beta_j|,$$

where at least some of the L_ν^j are different.

Let $[L_\nu]$ (or $[\sum_{j=1}^r L_\nu^j]$) be a dyadic rational number exceeding L_ν (or $\sum_{j=1}^r L_\nu^j$) and sufficiently close to the latter, and let $[\varepsilon_\nu]$ be the maximum dyadic rational number that is less than or equal to ε_ν and whose numerator is the same as that of $[L_\nu]$ (or $[\sum_{j=1}^r L_\nu^j]$). A dyadic number is a number of the form $p/2^m$, where p and m are natural numbers.

Theorem 2.1 *If criteria $\Phi_\nu(\alpha)$ are continuous and satisfy either the Lipschitz condition or the special Lipschitz condition, then to approximate $\Phi(D)$ to within an accuracy of ε it is sufficient to have*

$$\max_\nu 2^\tau \left(\frac{[L_\nu]}{[\varepsilon_\nu]}\right)^r \quad \text{or} \quad \max_\nu 2^\tau \left(\frac{[\sum_{j=1}^r L_\nu^j]}{[\varepsilon_\nu]}\right)^r$$

points of the P_τ *net.*

The proof is given in [21]. We note that τ increases with r. For details on τ, see [22].

Note that similar estimates for problems of finding the absolute extremum of functions satisfying the Lipschitz condition have been obtained in a number of works by alternative methods. For example, a similar estimate was obtained in [23].

The number of points needed to calculate the performance criteria in this estimate may be so large that the speed of present-day computers may prove to be inadequate. This difficulty may be overcome by developing “fast” algorithms dealing not with an entire class of functions but instead taking into account the features of the functions of each concrete problem.

To approximate a feasible region $\Phi(D)$, such an algorithm may be constructed in the following way. (Although all subsequent considerations presume that the Lipschitz condition is satisfied, they are valid as well for the special Lipschitz condition if constant L_ν is replaced by $\sum_{j=1}^{r} L_\nu^j$). Let the Lipschitz constants L_ν, $\nu = 1, \dots, k$, be specified, and let N_1 be the subset of the points from the set D that are either the Pareto optimal points or lie within the ε-neighborhood of a Pareto optimal point with respect to at least one criterion. In other words, we have the following: $\Phi_\nu(\alpha^0) \le \Phi_\nu(\alpha) \le \Phi_\nu(\alpha^0) + \varepsilon_\nu$, where $\alpha^0 \in P$, and P is the Pareto optimal set. Also, let $N_2 = D \backslash N_1$ and $\overline{\varepsilon_\nu} > \varepsilon_\nu$ where $\overline{\varepsilon_\nu}$ is a certain number defined in proving Theorem 2.2.

Definition 2.1 A feasible solution set $\Phi(D)$ is said to be normally approximated iff any point of set N_1 is approximated to within an accuracy of ε, and any point of set N_2 to within an accuracy of $\overline{\varepsilon}$.

Theorem 2.2 *There exists a normal approximation $\Phi(D_{\boldsymbol{\varepsilon}})$ of a feasible solution set $\Phi(D)$* [24].

2.9 The Pareto Optimal Set Approximation [2, 6]

Since the Pareto optimal set is unstable, even slight errors in calculating criteria $\Phi_\nu(\alpha)$ may lead to a drastic change in the set. This implies that by approximating a feasible solution set with a given accuracy we cannot guarantee an appropriate approximation of the Pareto optimal set. Although the problem has been tackled since the 1950s, a complete solution acceptable for the majority of practical problems is still to be obtained. Nevertheless, promising methods have been proposed for some classes of functions [25–27].

Let P be the Pareto optimal set in the design variable space; $\Phi(P)$ be its image; and ε be a set of admissible errors. It is desirable to construct a finite Pareto optimal set $\Phi(P_{\boldsymbol{\varepsilon}})$ approximating $\Phi(P)$ to within an accuracy of ε.

Let $\Phi(D_{\boldsymbol{\varepsilon}})$ be the ε-approximation of $\Phi(D)$, and $P_{\boldsymbol{\varepsilon}}$ be the Pareto optimal subset in $D_{\boldsymbol{\varepsilon}}$. As has already been mentioned, the complexity of constructing a finite approximation of the Pareto optimal set results from the fact that, in general, in approximating the feasible solution set $\Phi(D)$ by a finite set $\Phi(D_{\boldsymbol{\varepsilon}})$ to within an accuracy of ε, one cannot achieve the approximation of $\Phi(P)$ with the same accuracy. Such problems are said to be ill-posed in the sense of Tikhonov [28]. Although this notion is routinely used in computational mathematics, let us review it here.

Let P be a functional in the space X, $P : X \to Y$. We suppose that there exists a solution $y^* = \inf P(x)$, and $V_\varepsilon(y^*)$ is the neighborhood of the desired solution y^*. Now let us single out an element x^* (or a set of elements) in space X and its δ-neighborhood $V_\delta(x^*)$ and call x_δ^ε a solution to the problem of finding the extremum of P if the solution simultaneously satisfies the conditions $x_\delta^\varepsilon \in V_\delta(x^*)$ and $P(x_\delta^\varepsilon) \in V_\varepsilon(y^*)$. If at least one of the conditions is not satisfied for arbitrary values of ε and δ, then the problem is called ill-posed (in the sense of Tikhonov).

An analogous definition may be formulated for the case when P is an operator mapping space X into space Y. Let us set $X = \{\Phi(D_\varepsilon), \Phi(D)\}$; $Y = \{\Phi(P_\varepsilon), \Phi(P)\}$, where $\varepsilon \to 0$, and let $P : X \to Y$ be an operator relating any element of X to its Pareto optimal subset. Then, in accordance with what was said before, the problem of constructing sets $\Phi(D_\varepsilon)$ and $\Phi(P_\varepsilon)$ belonging simultaneously to the ε-neighborhoods of $\Phi(D)$ and $\Phi(P)$, respectively, is ill-posed. Of course, in the spaces X and Y, the metric or topology [29] that corresponds to the system of preferences on $\Phi(D)$ must be specified.

Let us define the V_ε-neighborhood of a point $\Phi(\alpha^0) \in \Phi(\Pi)$ as

$$V_\varepsilon = \left\{\Phi(\alpha) \in \Phi(\Pi) : \left|\Phi_\nu\left(\alpha^0\right) - \Phi_\nu(\alpha)\right| \leq \varepsilon_\nu, \nu = 1, \ldots, k\right\}.$$

In Theorem 2.3 provided below, we construct a Pareto optimal set $\Phi(P_\varepsilon)$ in which for any point $\Phi(\alpha^0) \in \Phi(P)$ and any of its ε-neighborhoods V_ε there may be found a point $\Phi(\beta) \in \Phi(P_\varepsilon)$ belonging to V_ε. Conversely, in the ε-neighborhood of any point $\Phi(\beta) \in \Phi(P_\varepsilon)$, there must exist a point $\Phi(\alpha^0) \in \Phi(P)$. The set $\Phi(P_\varepsilon)$ is called an approximation possessing property M. Let $\Phi(D_{\boldsymbol{\varepsilon}})$, an approximation of $\Phi(D)$, have been constructed.

Theorem 2.3 *If the conditions of Theorem 2.1 are satisfied, then there exists an approximation $\Phi(P_{\boldsymbol{\varepsilon}})$ of Pareto set $\Phi(P)$ possessing the M-property.*

The theorem has been proved by analyzing the neighborhoods of the so-called "suspicious" points from $\Phi(D_{\boldsymbol{\varepsilon}})$, that is, the points with neighborhoods that contain the true Pareto optimal vectors. If we find new Pareto optimal vectors in the neighborhoods of the "suspicious" points, then these vectors may be added to $\Phi(P_\varepsilon)$. Taken together with $\Phi(P_\varepsilon)$, they form the ε-approximation of a Pareto optimal set, [30].

It is shown in [6] that this approach solves the problem of the ill-posedness (in the sense of Tikhonov) of the Pareto optimal set approximation.

3 Multicriteria Identification. Adequacy of Mathematical Models

Multicriteria identification is a new direction that is of great value in applications. In the past we have solved a number of optimization problems. To a considerable extent, our success in this was owed to the adequacy of the corresponding mathematical models.

In the most common usage, the term "identification" means construction of the mathematical model of a system and determination of the parameters α_j (design variables) of the model by using the information about the system response to known external disturbances. Very often, when solving identification problems, the expert has no information about the limits α_j^* and α_j^{**} for many of the variables.

As a rule, these applied identification problems have been treated as single-criterion problems (see, e.g., [31, 32]). In the majority of conventional problems, the system is tacitly assumed to be in full agreement with its mathematical model. However, for complex engineering systems (e.g., machines) we generally cannot assert a sufficient correspondence between the model and the object. This does not allow one using a single criterion to evaluate the adequacy. In multicriteria identification problems, there is no necessity of artificially introducing a single criterion to the detriment of the physical essence of the problem. When constructing a mathematical model, one first defines the class and structure of the model operator, that is, the law according to which the disturbances (input variables) are transformed into the system response (output processes). This is called structural identification. For mechanical systems, structural identification means determining the type and number of equations constituting the mathematical model of the system. Structural identification is necessary if there is no preliminary information about the structure of the system or this information is not sufficient for compiling equations.

Parametric identification is reduced to finding numerical values of the equation coefficients, based on the realization of the input and output processes. In doing so, frequency responses, transfer functions, and unit step functions are often used. A number of problems require preliminary experimental determination of the basic characteristics of a mechanical system (e.g., the frequencies, shapes, and decrements of natural oscillations). When solving optimization problems, we have used the concept of performance criterion. In identification problems, we deal with particular adequacy (proximity) criteria. By adequacy (proximity) criteria we mean the discrepancies between the experimental and computed data, the latter being determined on the basis of the mathematical model.

For example, when identifying the parameters of the dynamical model of an automobile, it is necessary to take into account such important indices (particular criteria) as vibration accelerations at all characteristic points of the driver's seat, driver's cab, frame, and engine; vertical dynamical reactions at contact areas between the wheels and the road; relative (with respect to the frame) displacements of the cab, wheels, engine, etc. [2, 3].

In all basic units of the structure under study, we experimentally measure the values of characteristic quantities of interest (e.g., displacements, velocities, accelerations, etc.). At the same time, we calculate the corresponding quantities by using the mathematical model. As a result, particular adequacy (proximity) criteria are formed as functions of the difference between the experimental and computed data. Thus we arrive at a multicriteria problem. The multicriteria consideration makes it possible to extend the application area of the identification theory substantially.

3.1 Parameter Space Investigation Method in Problems of Multicriteria Identification

We denote by $\Phi_\nu^c(\alpha)$, $\nu=\overline{1,k}$ the indices (criteria) resulting from the analysis of the mathematical model that describes a physical system, where $\alpha=(\alpha_1,\dots,\alpha_r)$ is the vector of the parameters of the model. Let $\Phi_\nu^{\exp}$ be the experimental value of the νth criterion measured directly on the prototype. The experiment is assumed to be sufficiently accurate and complete. By completeness we mean that the criteria $\Phi_\nu^{\exp}$ are measured in all basic units at the most characteristic points of the structure. The amount of measurement data must be sufficient for correct formulation of the identification problem.

Suppose there exists a mathematical model describing the behavior of the system. Let $\Phi=(\|\Phi_1^c-\Phi_1^{\exp}\|,\dots,\|\Phi_k^c-\Phi_k^{\exp}\|)$, where $\|\cdot\|$ is a particular adequacy (closeness, proximity) criterion. This criterion, as has already been mentioned, is a function of the difference (error) $\Phi_\nu^c-\Phi_\nu^{\exp}$. Very often it is given by $(\Phi_\nu^c-\Phi_\nu^{\exp})^2$ or $|\Phi_\nu^c-\Phi_\nu^{\exp}|$. If the experimental values $\Phi_\nu^{\exp}$, $\nu=\overline{1,k}$ are measured with considerable error, then the quantity $\Phi_\nu^{\exp}$ can be treated as a random variable. If this random variable is normally distributed, the corresponding adequacy criterion is expressed by $M\{\|\Phi_\nu^c - \Phi_\nu^{\exp}\|\}$, where $M\{\|\cdot\|\}$ denotes the mathematical expectation of the random variable $\|\cdot\|$. For other distribution functions, more complicated methods of estimation are used, for example, the maximum likelihood method.

We formulate the following problem by comparing the experimental and computational data, determining to what extent the model corresponds to the physical system, and finding the variables of the model. In other words, it is necessary to find the vectors α^i satisfying conditions (1) and (2) and, in addition, the inequalities

$$\left\|\Phi_\nu^c\left(\alpha^i\right) - \Phi_\nu^{\exp}\right\| \leq \Phi_\nu^{**}. \tag{6}$$

Conditions (1), (2), and (6) define the feasible solution set D_α. Here, Φ_ν^{**} are criteria constraints that are determined in the dialogue between the expert and a computer. To a considerable extent, these constraints depend on the accuracy of the experiment and the physical sense of the criteria Φ_ν.

3.2 The Search for the Identified Solutions

The formulation and solution of the identification problem are based on the parameter space investigation method. In accordance with the algorithm given above, we specify the values Φ_ν^{**} and find vectors satisfying conditions (1), (2), and (6). The vectors α_{id}^i belonging to the feasible solution set D_α will be called *adequate vectors*.

The reconstruction of the parameters of a specific model on the basis of (1), (2), and (6) is the main purpose and essence of multicriteria parametric identification. Having performed this procedure for all structures (mathematical models), we thus carry out multicriteria structural identification.

The vectors α_{id}^i that belong to the set of adequate vectors and have been chosen by using a special decision making rule will be called *identified vectors*.

The role of the decision making rule is often played by informal analysis of the set of adequate vectors. If this analysis separates several equally acceptable vectors α_{id}^i, the solution of the identification problem is non-unique.

The identified vectors α_{id}^i form the identification domain $D_{\mathrm{id}} = \bigcup_i \alpha_{\mathrm{id}}^i$. Sometimes, by carrying out additional physical experiments, revising constraints Φ_ν^{**}, etc., one can reduce the domain D_{id} and even achieve the result that this domain contains only one vector. Unfortunately, this is far from usual. Non-unique reconstruction of variables is a recompense for the discrepancy between the physical object and its mathematical model, incompleteness of physical experiments, etc.

If a mathematical model is sufficiently good (i.e., it correctly describes the behavior of the physical system), then multicriteria parametric identification leads to a nonempty set D_α. The most important factors that can lead to an empty D_α are imperfection of the mathematical model and lack of information about the domain in which the desired solutions should be searched for.

The search for the set D_α is very important, even in the case where the results are not promising. It enables the expert to judge the mathematical model objectively (not only intuitively), to analyze its advantages and drawbacks on the basis of all proximity criteria, and to correct the problem formulation.

Thus, multicriteria identification includes the determination and informal analysis of the feasible solution set D_α with regard to all basic proximity criteria, as well as finding identified solutions α^i_{id} belonging to this set.

Multicriteria identification is often the only way to evaluate the quality of the mathematical model and, hence, to optimize this model. The algorithm is successfully used in practice. Below we discuss some important problems that are solved by using this algorithm.

3.3 Operational Development of Prototypes

We will discuss the problems of perfecting engineering systems (machines). These problems are mainly related to the operational development of a prototype of a machine designed for serial and mass production. First, the machine is tested. The structure of the test is determined by the type of the machine (an airplane, car, ship, machine tool, etc.). For example, cars are subjected to laboratory (bench) tests including strength, fatigue, and vibration investigations of both individual units and the car as a whole.

Significant attention is paid to road tests. These are carried out on proving grounds where the car is tested on properly profiled road sections in different conditions depending on the load carried and the speed of the car. Apart from this, cars are tested on ordinary roads under conditions close to operational ones. Thus, cars are subjected to bench and road tests. These tests are aimed at detecting imperfections with subsequent operational development of the prototype so as to satisfy the customer's demands. Operational development is aimed at increasing durability and reliability, reducing vibrations and noise, etc.

It is of great importance to make the process of operational development as short as possible. This is the main problem faced by experts of cars and other machines. We suggest carrying out the prototype's operational development in two stages. In the *first stage*, accelerated tests (for instance, bench tests) are performed. These tests allow identifying the mathematical model of the object and determining its parameters.

In the *second stage*, after multicriteria identification, the expert formulates and solves a multicriteria optimization problem. In doing this, he uses the mathematical model, whose adequacy was established in the first stage. Based on the results of the optimization, the improvement of the prototype is carried out, and then the tests are reproduced. This cycle is repeated until the expert decides to terminate the operational development.

Thus, in the first stage, the set D_α is found as a result of multicriteria identification. In the second stage, the optimization problem is solved: we construct the parallelepiped Π in D_α, determine the vector of performance criteria, and find the feasible solution set D.

We have already mentioned problematic application of optimization in design problems with significant discrepancy between the mathematical model and the physical system, as well as with improperly specified constraints. The results of optimization in such cases are often of no practical value. According to our approach, we obtain a validated model and the set D_α resulting from the multicriteria identification. This, to a sufficient extent, justifies the optimization performed in the second stage and substantiates the recommendations for improving the prototype of a machine. In addition, this approach is expected to significantly reduce the amount of expensive and time-consuming tests in the course of the operational development of objects [33–36].

4 Multicriteria Optimization of Large-Scale Systems

When designing objects, one has to deal with complex mathematical models. It is typical that these models have many hundreds of degrees of freedom, are described by high-order sets of equations, and, as has already been mentioned, the calculation of one solution can take significant computer time. This implies that it is not always possible to solve problems such as (1)–(3) directly [1, 2, 37]. In some cases, stating and solving these optimization problems requires high-performance computers with large memory. The software package MOVI allows us to tackle computationally expensive problems in *parallel or distributed mode*, so that the desired number of trials N is distributed among k computers. Thus, each computer finds a feasible solution set for its own subproblem. Next, all feasible solution sets are combined and a single Pareto optimal solution set is constructed. If this approach does not solve the problem, other techniques should be proposed. One remedy may be to split (decompose) a large-scale system into subsystems that can be easily optimized, and then aggregate the optimization results for each subsystem to obtain near-optimal solutions for the whole system. This will allow the expert to determine the requirements for the subsystems so as to make a machine optimal as a whole and, in this way, justify the proposals for designing different units of the object.

4.1 Decomposition and Aggregation of Large-Scale Systems

To solve this problem we can use an approach associated with considering the whole system as a hierarchical structure [2, 3]. The lower level of this structure comprises subsystems, whereas the higher level is the system as a whole. In many cases, the optimization can be done more simply at the lower level. Therefore, by using the results of the optimization at the lower level and thus reducing the number of competing solutions for the whole system, we can optimize the system in reasonable time. This approach was proposed relatively recently, and only the first steps have been made in this direction. In particular, this is true for the methods proposed here. Nevertheless, the results obtained can be used to optimize many large-scale systems.

Since the proposed approach is based on the optimization of the whole system through the optimization of its subsystems, we briefly describe the relation between the criteria for the system and subsystems. There are three possibilities for this relation:

(A) Some of the criteria of a subsystem can implicitly affect the performance criteria of the system as a whole, and very often, such subsystem criteria are absent from the list of performance criteria of the whole system. This situation is typical for the majority of complex engineering systems.
(B) Some of the system criteria cannot be calculated at the subsystem level.

(C) There are criteria that can be calculated for both the whole system and its subsystems.

There are the following common features of these problems.

(1) Some of the mathematical models cannot be effectively optimized with respect to the whole criteria vector Φ, because it takes significant amount of computer time to formulate and solve the problem (1)–(3). However, the calculation of the values of particular performance criteria Φ_ν takes a reasonable amount of computation time.
(2) The system is "partitioned" into subsystems. The couplings connecting the subsystems will be called external. To separate out some of the subsystems as autonomous, it is necessary to analyze the interaction of this subsystem with all other subsystems, as well as the external disturbances applied to the subsystem by the environment.
(3) The subsystems can be optimized. The idea of optimizing the whole system consists of the following. First, when optimizing each (ith) subsystem, we obtain for this subsystem a pseudo-feasible solution set $\overline{D}^i$, which, as a rule, is somewhat larger than the true feasible solution set. Next, we compile the vectors for the whole system using the respective vectors from the sets $\overline{D}^i$. Then we check whether the criteria and functional constraints of the system are satisfied in this domain and, as a result, obtain the feasible solution set D for the whole system. Finally, we search for the optimal solution over the set D. In other words, optimizing the whole system is reduced, to a considerable extent, to the optimization of its subsystems.

Algorithms that implement this idea, take into account (i) different dependences between the design variables of the system and its subsystems, (ii) the basic possibilities of simplifying the initial mathematical model, and (iii) the ways of determining external disturbances for subsystems, etc.

5 Applications

The PSI method has been created and successfully integrated into various fields of industry, science, and technology. This method has been used in designing the space shuttle, nuclear reactors, unmanned vehicles, airplanes, cars, pumping units, ships, metal tools, bridges, wind power system, wireless battlefield networks, energy efficient sensor networks, and robots. The relevant bibliography is partly presented in [1–5]. Several brief examples are provided below.

Example 5.1 (Multicriteria Identification of Characteristics of a Spindle Unit and Its Operational Development [1, 2, 38]) The precision, reliability, and chatter stability of metal-cutting machines depend strongly on the characteristics of their spindle units. This calls on conducting the dynamic and thermal analyses of the spindle units in the design and experimental development stages, using the data obtained by testing a prototype. To solve the problem of identification, mathematical models of the dynamic and thermal systems of the spindle unit were developed on the basis of the finite element method.

Identification of Parameters of the Models Since inertial and stiffness parameters of a spindle can be easily expressed through its geometric dimensions, the values of the stiffness coefficients and the damping factors of the supports are the parameters to be identified in the dynamical model. The parameters to be identified in the thermal model are the values of the coefficients of convective heat removal from the spindle open surface, the spindle surface between the front and rear supports, and the spindle casing surface.

In the first problem, the set of criteria determining the static and dynamic behavior of the spindle over the range of frequencies from 0 to 600 Hz (criteria $\Phi_1 - \Phi_{13}$) was determined by analyzing the static elastic line of the spindle and its first shape of vibration (345 Hz). In the second problem, it is reasonable to estimate the correspondence between the thermal model and the actual thermal characteristics of the spindle unit according to the temperature values at the structure points (criteria $F_1 - F_{23}$).

The adequacy criteria $\Phi_\nu, \nu = \overline{1, 13}$ were defined. In the same way, we determine $F_\nu, \nu = \overline{1, 23}$. Usually, a model is considered to be adequate to the object in the νth criterion, if the relative discrepancy does not exceed 20 %, see [38].

Solution of the Identification Problems We performed 512 trials in each of the initial parallelepipeds Π_1^1 and Π_2^1 corresponding to the dynamic and heat models, respectively. The calculations yielded acceptable discrepancies in static (5–10 %) and dynamic (9–18 %) criteria. However, the discrepancies in thermal criteria turned out to be unacceptably large (34–49 %). Thus, the correctness of the specification of the limiting variation in the convective heat transfer coefficients of the thermal model was doubtful.

Further modifications of the boundaries of the parallelepiped Π_2^1 for the thermal model were essentially based on analyzing histograms of feasible solutions and graphs of criterion vs. design variable and criterion vs. criterion. As a result, we determined the boundaries of a new parallelepiped Π_2^2. After we refined the heat exchange coefficients, the discrepancies with respect to the thermal criteria did not exceed 19 %.

For adequacy criteria Φ_ν^{**} and F_ν^{**} not exceeding 20 %, we determined the feasible sets D_α for the dynamic and thermal models, which consisted of four and six vectors, respectively. After an informal analysis, the vector $\alpha_{\mathrm{id}}^{41}$ for the dynamic model and the vector $\alpha_{\mathrm{id}}^{90}$ for the thermal model were adopted as the best ones. These vectors were chosen for the following reasons:

1. The vector $\alpha_{\mathrm{id}}^{41}$ corresponds to minimum discrepancies with respect to the static displacement Φ_2 (6 %) and the dynamic displacement Φ_3 (11 %). These displacements are known to influence the precision of machining most strongly.
2. The vector $\alpha_{\mathrm{id}}^{90}$ most accurately reflects the thermal state of the spindle unit near the supports (the temperature discrepancies do not exceed 13 %), which is of great practical importance for properly choosing the type of lubricant.

Thus, having constructed reliable mathematical models, we can proceed to the next stage of experimental development of the spindle, and then to its optimization.

Solution of the Optimization Problem To determine the feasible sets for the design variables, we analyzed the adequate vectors and constructed parallelepipeds

$\overline{\Pi}_1 \subseteq \Pi_1^1$ and $\overline{\Pi}_2 \subseteq \Pi_2^2$ for the dynamic and the thermal models, respectively. Six performance criteria were selected to be minimized. These criteria characterize the static and dynamic stiffness of the spindle unit and the thermal state of its supports. 256 trials were performed in $\overline{\Pi}_1$ and $\overline{\Pi}_2$. Optimization was accomplished by varying the support stiffness coefficients and the heat exchange coefficients. The calculations revealed a set consisting of three Pareto optimal vectors of the spindle unit. In our case, the vector α^{114} was preferred, since it is close to the prototype as far as the temperature criteria are concerned and exceeds it in the static and dynamic stiffness by 8 % and 12.5 %, respectively. The results of the experimental development of the spindle unit prototype with the optimal design variables were checked experimentally and proved to be true to an acceptable accuracy of measurements.

Example 5.2 (Problem of a Naval Ship Design [1, 18, 19]) Among the particular features of this problem are the high dimensionality of the design variable vector (45 design variables) and the difficulties in improving a reasonably good prototype under strong constraints on six performance criteria (the propulsion power factor; the electrical power factor; the volume factor, the region factor, the weight factor, and the cost), nine pseudo-criteria, and seven functional dependences. Since calculation of one criterion vector took less than 1 second, hundreds of thousands of trails were conducted in several experiments. Each subsequent experiment was carried out on the basis of the previous one. In the process of solving, experts corrected the problem six times. Interestingly, after the first two corrections of the initial formulation of the problem, the feasible set was empty. However, the histograms and graphs showed how to modify the constraints. This allowed us to improve the prototype significantly.

Example 5.3 (Problem of Design of a Car for Shock Protection [2, 39]) The criteria in this problem are based on finite element model with thousands of elements and nodes, and it takes approximately two hours to compute one vector of criteria. This problem has 10 criteria: the mass of structure and residual strains in the car body after impact in the nine most dangerous points of the rear panel. The number of design variables in this problem is 13. However, since it requires such a large amount of time to compute one vector of criteria, the optimization of design variables is very difficult to implement. Therefore, the initial model was decomposed into two models, the model of the bumper and model of the rear panel. In the *first step*, it was necessary to define the set of feasible parameters for these subsystems. For this purpose, 300 tests were carried out for each subsystem. Each test required no more than 10 minutes. In the aggregation procedure (*the second step*), design variables of each of the feasible design variable vectors of the bumper are united (concatenated) with design variables of each of the corresponding design variable vectors of the rear panel. For all solutions, we calculated criteria related to the whole system. The feasible solution set appeared to be empty, i.e., the prototype cannot be improved. Therefore the problem has been reformulated (i.e., the designs of the bumper and the rear panel were modified by introducing additional stiffening ribs). As a result, 15 consistent design variable vectors were defined. For these vectors we calculated all criteria using the initial model. The number of feasible solutions satisfying all constraints of the structure was nine. The number of Pareto optimal solutions was five.

The expert preferred the structure using design variable vector 257 of the bumper and design variable vector 181 of the rear panel. A more uniform distribution of residual displacements (compared with other designs) and an acceptable mass are features of this structure. As a result, we have obtained a new design that satisfied all mass, residual strain, and shock requirements.

Example 5.4 (Problem of Design of the Flight Control System [1, 40]) This example presents preliminary results of the application of the PSI method and MOVI software system to the problem of design of the $\mathcal{L}_1$ flight control system implemented in the two turbine-powered dynamically scaled Generic Transport Model, which is a part of the Airborne Subscale Transport Aircraft Research aircraft at the NASA Langley Research Center. The problem statement was corrected twice. For the purpose of economy of compute time in the first stage of the PSI method, the feasible solution set was constructed taking into account the so-called "fast" criteria (in the sense of the compute time required for one criteria vector). Then in the second stage, the "slow" criteria were added. In the first iteration, constraints on 14 criteria and pseudo-criteria were defined and 7 criteria were optimized; 1,024 trails were performed, and it took approximately 1 minute to compute one vector of criteria and pseudo-criteria. In the second iteration, we had 16 criteria and pseudo-criteria. Constraints on the most criteria and pseudo-criteria were defined and 9 criteria were optimized; 512 trails were performed, and it took approximately 10 minutes to compute one vector of criteria and pseudo-criteria. While 20 solutions were found to be feasible after the first iteration, this number was reduced to 6 feasible solutions (all Pareto optimal) after the second iteration. As a result, the optimal design has significantly improved upon the basic characteristics of the adaptive flight control system.

6 Conclusions

One of the key features of engineering optimization problems is that their solutions are based on construction and analysis of the set of feasible solutions. This is due to the fact that experts cannot correctly specify the constraints, primarily the criteria constraints. It is only the results of the investigation of the design variable space and the criterion space that enable an expert to define the feasible solution set. In order to construct the feasible solution set, a method called Parameter Space Investigation (PSI) has been created and successfully integrated into various fields of industry, science, and technology. In the PSI method, stating and solving the problem is a single interactive process. In the course of investigation of the feasible solution set, the experts are estimating the effect of the constraint correction. Usually the multicriteria analysis shows expediency of correction of the initial statement of the problem.

Based on the PSI method, the methods for approximation of the feasible solution set and Pareto set, identification of the parameters of mathematical models, decomposition and aggregation of large systems were developed, and are summarized in this paper.

In all cases that we have encountered in our experience, the results surpassing prototypes in all criteria or in the most important ones were obtained. These results are due to application of the MOVI software system that allows one constructing test tables and tables of feasible and Pareto solutions. These tables together with various visualization tools (histograms and graphs) allow one correcting the statement of the problem and improving values of the basic criteria.

Construction and approximation of the feasible solution set in identification and optimization of large-scale systems are challenging problems, and their solution is vital for the field of engineering optimization and other sciences. The present work made a contribution to this exciting problem area by describing our results in developing methods for construction and approximation of the feasible solution set. While the methods can be readily applied to solve many types of real-world problems, there are open problems that require further research. For example, these are: problems of vector identification of mathematical models that are constructed from observational data (these problems are common to biology and medicine); multicriteria problems of information transfer with very-high dimensional vectors of design variables (these problems are relevant for cellular communication); and multicriteria optimization with discrete space of design variables.

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